REVIEW OF ADVANCES IN MATHEMATICAL MODELS FOR STRUCTURAL BEHAVIOR OF BITUMINOUS AND SIMILAR MEMBRANE SYSTEMS

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At the International Flat Roof Conference in Brighton, England, August 1984, William Cullen outlined the findings of the NRCA's 1983 Project Pinpoint.[1] It was reported that almost 20 percent of problem jobs related to membrane splitting failures. Some 48 percent of problem jobs related to a combination of physical effects in which splitting could well have been involved.

The case for a better understanding of the structural behavior of single or multi-layer waterproofing systems rests not only on the obvious need to deal with the risk of splitting by informed design, but also on other desirable benefits which naturally accrue from a proper understanding of the subject, namely the opportunity to:

- design and develop cost effective membrane systems and materials
- improve the relevance and methods of system assessment and testing, and
- prepare the proper foundation for an engineering approach to roofing performance.

WHY MATHEMATICAL MODELS?

The purpose of a mathematical model is to describe the interdependence and collective action of an assembly of materials, whose properties, dimensions and combinations are prescribed. It has to be demonstrated that any given model does not itself lead to contradictions or to physically impossible conclusions, and that it is capable of being used to predict with acceptable accuracy the outcome of a wide range of controlled experiments.

The value of any model resides in its ability to predict observable phenomena. System performance tests strictly require to be based on some coherent idea of system action as described by a proven model.

MODEL FOR BUILT-UP ROOFING SYSTEM

An understanding of the structural behavior of a waterproofing membrane is obtained by regarding the membrane as a flat plate bonded or otherwise connected to a deformable substrate.

The traditional built-up membrane, whether composed of improved materials or not, is a bitumen reinforced plate, with a layer of reinforcement corresponding to each of the superimposed roofings of which the membrane is composed. A flexible plastics sheeting may be interposed in place of a reinforcing fabric without invalidating the form or generality of the model.

The equations relating to the mechanical behavior of a multi-layer membrane system derive from a consideration of (a) the equilibrium of forces acting on a small and arbitrary element of reinforcing fabric, and (b) the physical continuity of the medium from which the elements and their immediate surroundings are formed.

The mathematical steps relating to a single-layer membrane (one reinforcing fabric only) are given in the references.[2,3]

The corresponding equation for a fabric embedded within a built-up layer and interposed between other layers of reinforcing fabric contains an additional term not present in the single-layer case. The additional term allows for the shear stresses generated by the reinforced bitumen layers above the fabric and complements those generated by the layers beneath it. Clearly, the additional term does not apply to the upper most layer of any multi-layer system.

The following terms relate to the i^{th} layer of reinforcement:

Let	$u_i(x)$	=	displacement of a point x from its original position
	E_i	=	elastic (hookean) modulus
	h_i	=	thickness of bituminous layer separating the i^{th} fabric from the nearest one beneath it
	m_i	=	ratio h_i/h_{i+1}
	G_i	=	shear modulus of bitumen (or adhesive)
	a_i	=	G/E_ih_i
	D	=	differential operator $\delta/\delta x$

The convention is that $i=1$ relates to the first fabric located immediately above the substrate to which the membrane is bonded.

The displacement $u_o(x)$ therefore relates to the relative movement in the substrate surface itself, which movement is an external cause of membrane deformation.

The equation representing the physically compatible equilibrium of an element of fabric is:

$$a_iu_{i-1} + \{D^2 - a_i(1+m_i)\}u_i + a_im_iu_{i+1} = o \qquad (1)$$

The equation holds even when there are temperature fluctuations and shrinkage in the fabric, provided that such effects are independent of the position co-ordinate x. Where temperature and shrinkage vary in the fabric from point to point, the right hand side of (1) becomes $D(\beta_i\theta - \zeta_i)$

here β_i	=	coefficient of expansion of i^{th} fabric
$\theta(xt)$	=	fabric temperature
$\zeta_i(xt)$	=	i^{th} fabric shrinkage

The strain in the fabric arising from mechanical stress is given by:

$$\epsilon_i(x) = Du_i - \beta_i\theta + \zeta_i \quad (2)$$

Note that Du_i is *observable* strain which embraces the mechanical strain ϵ_i and the temperature and shrinkage effects, if any.

A built-up system comprising n superimposed reinforcing fabrics therefore generates n second order differential equations, each given by eq(1) and obtained by successively writing i = 1, 2, 3.....n. It is convenient, for the purposes of notation, to have the only term involving u_o on the right hand side of the set of equations so formed.

A n x n system matrix $A_{\mu\mu}$ is formed from the array of coefficients of u_i. It also is appropriate to regard u_i as a component of a system *displacement* vector u_α.

Fabric displacements from an initial position of equilibrium are the result of effects represented by terms appearing on the right hand side of the equation. For notational convenience, such terms will be represented by v_i. Together they constitute an *excitation* or *stress generating* vector, v_α. where the subscript α refers to the excitation pertaining to the α^{th} layer.

The action of an n-layer built-up system is represented symbolically in matrix notation by:

$$A_{i\mu}u_\mu = v_i \quad (3)$$

where	μ		is summed from 1 to n
	i		is any number chosen from 1 to n
and	v_i	=	$-a_1u_0\delta_i^1 + \beta_iD\theta - D\zeta_i$
with	δ_i^1	=	Kronecker's delta ($\delta_1^1 = 1$, $\delta_i^1 = o$, $i > 1$)

The last of these states that $v_1 = -a_1u_0 + \beta_1D\theta - D\zeta_1$ and $v_i = \beta_iD\theta - D\zeta_i$ for all values of i greater than 1. Where temperature and shrinkage do not vary with position,

$$v_1 = -a_1u_0 \text{ and } v_i = o \text{ for } i > 1.$$

Eq(3) is the membrane system equation as written in its most concise and general form, in which external effects, represented by the v_i vector, give rise to displacements u_r (and therefore to stresses) in each of the n fabrics. The connection between v_i and u_r is governed by the system matrix A, the elements of which are the coefficients in the reinforcing fabric equations typically given by eq(1).

It also should be noted that A_{rs} is an operational matrix involving the operator ($D^2 \equiv \delta^2/\delta x^2$) in the terms which form its diagonal trace.

For a single-layer system, the matrix reduces to a single term:

$$A = D^2 - a_1$$

For a two-layer system, there are four terms:

$$A = \begin{matrix} D^2 - a_1(1+m_1) & a_1m_1 \\ a_2 & D_2 - a_2 \end{matrix}$$

and for three-layers:

$$A = \begin{matrix} D^2 - a_1(1+m_1) & a_1m_1 & o \\ a_2 & D^2 - a(1+m_2) & a_2m_2 \\ o & a_3 & D^2 - a_3 \end{matrix}$$

MEMBRANE DISPLACEMENT FUNCTION

It is not the purpose of this paper to deal with the mathematical aspects of the membrane system equation.[3] It is sufficient to note that this differential equation can be transposed and solved to give the *membrane displacement* function in explicit form.

$$u_r(x) = A^{\mu r}\nu_\mu \div |A| + u_r^c(x) \quad (4)$$

where	ν_μ	=	the excitation vector, as previously defined: (μ summed from 1 to n)
	$\|A\|$	=	determinant of the system matrix A_{rs} of order n for an n-layer system
	A^{rs}	=	system determinant co-factor (obtained by striking out r^{th} row and s^{th} column in $\|A\|$)
	u_r^c	=	complementary function (satisfying the differential equation $A_{r\mu}u_\mu^c = o$)

The determinant $|A|$ will, and its cofactors A^{rs} may, contain terms involving the differential D^2. Thus, $A^{\mu r} \div |A|$ is a defined function of D^2 which operates on the elements of $\nu_\mu(x)$. The treatment of such mathematical operations is to be found in standard textbooks on differential equations.

It should be noted in passing that the first term on the right hand side of Eq(4) is called the *particular integral.* It represents the displacements in a section of membrane system resulting from excitations acting over the whole or part of that section. The second term, called the complementary function, gives the displacement of the section as a consequence of forces applied to or movements impressed at the extremities of the section. The complementary function deals with end effects.

In solving the membrane equation for a particular case it is necessary to choose an appropriate section of membrane. The conditions at the extremities of the section need to be known or otherwise specified. Either an extremity is unloaded and, therefore, free to be displaced, or it is fully fixed and able to develop a local stress.

The membrane displacement function must be adapted to these boundary conditions in order to obtain the correct solution for the specified case. The boundary conditions amount to a choice between:

(i)	$u_i(o)$	= o :	:	(restrained end at x = o)
or (ii)	Du_i	= $\beta_i\theta - \zeta_i$	:	(no load at x = o)

A further point of importance is that the displacement functions $u_i(x)$ are mathematically linear. This means that the effect of separate excitations are additive. The analyst therefore is able to investigate co-existing effects independently and more simply, and may choose to combine the separate results to obtain the corresponding net effect on the system. It also is possible to compare the importance of one effect against that of another.

MECHANICAL STRAIN IN MEMBRANE

The mechanical strain, $\epsilon_r(x)$, induced in the r^{th} layer of reinforcement at point x^r is given by eq(2).

$$\epsilon_r(x) = Du_r(x) - \beta_r\theta(x) + \zeta_r(x) \quad (2)$$

where $Du_r(x)$ is obtained by differentiating the membrane displacement function, eq(4), with respect to x.

If a section of membrane is chosen so that its extremities at x = o and x = L are deemed to be unloaded, the boundary conditions for the problem are $\epsilon_r(o) = \epsilon_r(L) = o$ for each and every value of r. The constants which appear in the complementary function are determined accordingly.

In general, the stresses will vary from point to point and from layer to layer at any given point, depending upon the form of excitation and boundary conditions. The distribution and partitioning of strains in two and three layer membrane systems have been derived by Koike.[4]

In general, the total force in the plane of a multi-layer membrane will be divided unequally between the superimposed layers of reinforcement. The load share carried by an unruptured layer will decrease according to its remoteness from the substrate. Initially, the first layer carries the greatest load share, and the top layer the least load share. The mechanism of membrane rupture usually will involve the sequential and separate rupture of each layer in turn.

The general action of a multi-layered system therefore is not unlike that of a number of superimposed single layers each acting in sequence with only relatively small assistance from its near neighbors. A consequence of this is that the practice of testing "dog bone" shaped specimens cut from a multi-layer build up does not give a proper indication of system strength. In practice, membranes are stressed by effects acting on their upper and lower surfaces and are therefore loaded eccentrically with respect to the membrane centroid. They are not loaded by axial forces applied at their extremities.

If mechanical and fatigue tests are to provide meaningful results, it is important to test membrane samples by bonding or otherwise attaching these to moving or reciprocating plates, rather than to grip sample ends in the jaws of a tensile testing machine.

SUBSTRATE FUNCTION $u_o(x)$

An important form of excitation arises from movement and strain in the substrate to which the membrane is attached, it being assumed that the attachment is capable of transmitting substrate movement to the membrane. The displacement of the substrate is given by the function $u_o(x)$ and this function appears in the excitation vector element ν_1.

The substrate function may be a continuous function of x, or it may be piecewise continuous with steps at specific points. Such steps or jumps, if any, are particularly important because they give rise to relatively high local stress concentrations in the membrane.

For example, one may consider substrate movement as a consequence of membrane temperature changes, θ. The substrate equation is then given by:

$$Du_o(x) = \beta_o\theta(x) \tag{5}$$

where β_o = coefficient of expansion of substrate.

The integration of (5) leads to a general expression for $u_o(x)$ given by:

$$u_o(x) = \beta_o \int \theta(x)dx + \Sigma g_\alpha S(x - x_\alpha) \tag{6}$$

Eq(6) not only contains the integral of the continuous temperature function, it also embraces possible discrete discontinuities or jumps in the magnitude of $u_o(x)$ at the specific locations, $x_1, x_2 \ldots$

The unit step function, $S(x - x_\alpha)$, is defined as zero when $x < x_\alpha$, and 1 when $x > x_\alpha$. Thus, at the point x_1, the displacement of the substrate is increased by a fixed amount g_1, and so on at other discontinuities, to form a series of increasing steps.

A butt joint between two substrate panels located at x_1 will constitute a potential discontinuity capable of being expressed mathematically by $g_1S(x - x_1)$.

The magnitude of g_1 will depend upon temperature and upon the location of fixed points in the substrate panels on either side of the butt joint. In general, g will increase as the temperature falls and decrease as it rises. The special characteristic of a discontinuity is that points in the substrate immediately on either side of the discontinuity may move simultaneously in opposing directions.

Other physical discontinuities are possible. For example, the premature rupture of a reinforcing layer will, at the point of separation, constitute a local discontinuity with the effect of producing a corresponding stress concentration in surviving layers.

The effects of discontinuities may be separately investigated by considering an excitation in the form of step functions only.

Thus temperature, which may generally be viewed as a continuous function, may give rise to substrate discontinuities. Temperature variations are more damaging because of the concentrated effect at discontinuities than they are as a result of temperature induced strains at points between them.

MECHANICAL STRAINS AT SUBSTRATE DISCONTINUITIES

Movement concentrated at a substrate discontinuity will give rise to a local stress concentration in each of the reinforcing layers bridging that discontinuity. The movement also will produce a locally high shear stress in the adhesive layer sandwiched between the substrate and the first layer equal to $gG/2h_1$.

The mechanical strain in the r^{th} layer may be expressed in the general form:

$$\epsilon_r = \tfrac{1}{2} g\lambda_r \sqrt{G/E_1h_1} \tag{7}$$

where
- g = movement at the discontinuity
- G = shear modulus of adhesive layers
- E_1 = modulus of first layer
- h_1 = distance between first layer and substrate
- λ_r = a correction factor specific to the r^{th} layer to allow for interaction of other layers

The fact that eq(7) relates to the properties of the first layer is arbitrary. The magnitude of the correction factor λ_r will vary according to the reference layer chosen, in this case the first layer.

Where the coefficients of thermal expansion of the membrane and substrate differ, there is an additional term in eq(7) of the order of $(\beta_o - \beta_r)\theta$ which gives the mechanical strain induced by differential thermal effects. However, its magnitude normally is very small by comparison with the principal term, and the temperature strain may safely be disregarded in most practical cases.

The correction factor λ_r is determined from eq(2) which is itself derived from the membrane displacements $u_r(x)$ calculated on the basis of isolated substrate discontinuities.

The following table gives the exact value of λ_n for the strain in the top layer of a single- or multi-layer membrane system. Last column in the table will be explained in section eight.

Ref No	No of Layers(n)	Moduli 1st	2nd	3rd(n)	Thickness	λ_n	λ_a
1	1	E	—	—	h	1.000	1.000
2	1	E	—	—	2h	0.707	0.707
3	1	E	—	—	3h	0.577	0.577
4	1	2E	—	—	3h	0.408	0.408
5	2	E	E	—	2h	0.447	0.577
6	2	E	2E	—	2h	0.358	0.477
7	2	2E	E	—	2h	0.383	0.500
8	2	E*	E	—	3h	0.357	0.447
9	3	E	E	E	3h	0.301	0.408
10	3	E	E	2E	3h	0.252	0.333

*First layer at 2h from substrate.

The λ_n values show the effect on the system of varying design parameters, such as the number of layers, moduli of reinforcing fabrics and total thickness.

System strain is reduced by one or a combination of:

(i) increasing total thickness

(ii) increasing number of reinforcing layers for a given thickness

(iii) increasing the modulus of one or more reinforcing layers

(iv) locating the stiffest layer uppermost

A reduction in the average shear modulus of the adhesive layers will independently reduce the corresponding strain. A 50 percent reduction in shear modulus will give a 70 percent reduction in strain.

The distribution of strain between layers of reinforcement is:

(a) **two-layer system** (moduli E,E)
first layer $\lambda_1 = 0.894$
second layer $\lambda_2 = 0.447$

(b) **three-layer system** (moduli E,E,E)
first layer $\lambda_1 = 0.871$
second layer $\lambda_2 = 0.388$
third layer $\lambda_3 = 0.301$

A SIMPLIFIED TREATMENT OF CORRECTION FACTOR λ

While the exact value for λ may be calculated quickly for two-layer systems,[5] the exact value for three-layer systems is disproportionately more time consuming to calculate.

An approximate and generally conservative estimate of λ_n for the top layer may be obtained by obsrving the following rules:

(i) The Eh product for a multi-layer system is taken as the sum of the separate products calculated for each individual layer. If a layer is deemed to have ruptured at the discontinuity, then that product is omitted from the sum.

(ii) The shear modulus of the adhesive layers separating the reinforcing fabrics is calculated from the average compliance $1/G = \Sigma h_i/G_i \div \Sigma h_i$ where h_i and G_i are as defined in section 3 above.

The values of λ so calculated, labelled λ_a, are given in the last column of the Table in the previous section. The value of λ overestimates the strain in the top layer by some 30 percent.

The value of λ_a will underestimate the top layer strain in cases where one or more reinforcing layers are deemed to be ruptured at the substrate discontinuity. The order of under estimate is 15 percent.

A practical membrane design methodology therefore would consist in transposing eq(7) to give:

$$g = (2\epsilon/\lambda_a) \quad \sqrt{E\,h\,/\,G} \tag{8}$$

The in-plane movement accommodating characteristic of a membrane then could be calculated according to eq(8) and so expressed as a g-factor.

The strain ϵ would require to be selected as the allowable strain in the top layer with due regard to the lesser of (i) the fatigue endurance of the top reinforcement and (ii) the fatigue endurance of the bituminous coating carried by that top layer.

A high fatigue endurance requires a material to be structurally underutilized. If $\bar{\epsilon}$ is the strain required to rupture the material in one application of a load, then the smaller the ratio $\epsilon/\bar{\epsilon}$, the greater the number of strain cycles capable of being accommodated before rupture.

Any two membrane systems therefore may be readily compared by looking at their respective g-factors as determined by eq(8).

A cautionary note is appropriate on the admissible modulus of a reinforcing fabric.

1. The modulus, which is given by the slope of the stress-strain line, has to be the value of that slope within the admissible range of strain.

   The modulus may be derived from load per unit width of fabric to produce a 2 percent elongation.

2. If a tensile test on a reinforcing fabric causes it to contract laterally while being elongated, the effective modulus will be greater in practice if lateral contraction is restrained.

3. If a fabric is composed of continuous strands embedded in a mat otherwise composed of randomly oriented fibres, no benefit necessarily will be derived from the continuous strands unless these are initially straight in the direction of loading.

   Furthermore, if the mat is able to rupture before the continuous strands do, the modulus will drop suddenly and the effectiveness of the reinforcement will be diminished accordingly. The allowable strain in the composite reinforcement should be limited to that of the weaker component, if the highest modulus is to be used as a design basis.

4. The modulus of most commonly available reinforcing fabrics tends to be directional. Prudent design would require the least modulus to be adopted. If not, the system performance will depend upon the direction of in-plane system excitations. Taking the greater modulus and disregarding the lesser value would give an incomplete and possibly misleading assessment of system capability.

ALLOWANCE FOR VISCO-ELASTIC BITUMEN

It has been tacitly assumed so far that the adhesive layer is composed of an elastic material with constant shear modulus G.

The assumption that bitumen behaves elastically is neither correct nor generally satisfactory as a realistic basis for mathematical modeling.

The rheological behavior of bitumen, as described by van der Poel, Huekelom and others, suggests that bitumen may be more accurately described as a thermo-rheologically simple Boltzmann body. This idealized rheological model for bitumen simultaneously admits both arbitrary bitumen temperature and loading histories, and gives the consequent strain history as a function of the previous two.

The essence of the model is that response functions (such as flow, relaxation or shear moduli), which depend upon loading time t and temperature θ, may be expressed in terms of a single temperature adjusted time variable ξ, known as the "reduced" time which then automatically allows for arbitrary temperature variations.

Thus, the shear modulus of bitumen under a constant stress $G(t,\theta)$ may be expressed as $\bar{G}(\xi)$, where the bar denotes a change of co-ordinates from $\{t,\theta\}$ to $\{\xi\}$ and where:

$$G(t,\theta) = \bar{G}(\xi) \qquad (9)$$

The co-ordinate transformation from t to ξ depends upon the temperature history of the bitumen $\theta(t)$ and is given by the shift function $\phi(\theta)$ according to:

$$d\xi = \phi\{\theta(t)\}dt, \text{ or} \qquad (10)$$

$$dt = d\xi + \phi\{\bar{\theta}(\xi)\} \qquad (11)$$

A shift function for bitumen, where deformation is governed mainly by the delayed-elastic component of flow, is proposed in reference.[6]

The temperature history may itself be expressed in physical time or in reduced time subject to:

$$\theta(t) = \bar{\theta}(\xi)$$

Variations in system temperature therefore are dealt with by expressing all time dependent variables in terms of reduced time. Bitumen temperature thus is relegated to the role of linking physical time and reduced time as though this was a factor related to the observer rather than one belonging to the membrane system under observation. Obviously, a given temperature history permits one to move from one time co-ordinate system to the other according to the transformations.[(10) or (11)]

A simplification of mathematical form is obtained by expressing[(9)] in terms of the image functions obtained by the Carson transform with respect to ξ of corresponding object functions. An asterisk will denote the transformation operation:

$$\bar{G}^*(p) = p\int_0^\infty e^{-p\xi}\bar{G}(\xi)d\xi$$

The general form of the system equations and the general or particular solutions remain unchanged by viewing the adhesive layer as a thermo-rheologically simple Boltzman body, provided that all time dependent excitations and moduli are expressed as the Carson transform of the corresponding object functions expressed in "reduced time" ξ.

For example, eq(7) for the strain produced by a joint opening g becomes more generally:

$$\bar{G}^* = \tfrac{1}{2}\bar{g}^*\lambda_r\sqrt{G^*/h_1E_1}$$

where $\bar{g}^*$ is the transform of $\bar{g}(\xi) = g(t)$

and $\bar{G}^*$ is the transform of $\bar{G}(\xi) = G(t,\theta)$

For a general purpose design method, the visco-elastic effects usually may be disregarded. For research and detailed mathematical investigation of practical phenomena, the disregard of visco-elastic effects generally will be grossly inaccurate if not misleading. For example, only by taking visco-elastic effects into consideration does one see why the repeated opening and closing of a substrate joint results in a rising compressive and diminishing tensile strain amplitude in the membrane. The corresponding case with a purely elastic adhesive layer reveals nothing but a cyclical tensile strain of constant amplitude.

Furthermore, cyclical substrate movement will cause the shear modulus of the visco-elastic adhesive layer to depend upon cycle time; the shorter the time the greater the modulus. A treatment of this case is given in[2].

CONCLUSIONS

1. A complete mathematical description of a single- or multi-reinforced system is both possible and manageable.
2. For the research worker, it provides a basis for understanding and for the design of experiments, provided account is taken of the visco-elastic effects, if any.
3. For the informed designer, the model provides a simplified design basis by means of which competing systems may be compared for cost effectiveness. Additionally, membranes may be designed to meet specified substrate movements, albeit with suitable safety factors to allow for the usual design imponderables.
4. For those concerned with Standards and testing, it provides a basis for the selection of tests and for material or system characterizations which are relevant and practically meaningful.

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